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Numerical Method for Navier-Stokes Equations
— Integral Formulation —

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An efficient difference algorithm is developed for solving partial differential equations. The essential point of this method is to construct a difference scheme not by numerical differentiation but by numerical integration. This is based on the well-known fact that numerical integration can be more accurate than numerical differentiation. Based on the integral forms of the original partial differential equations, weak solutions are obtained by this method. This enables us to capture the shock wave automatically. Although this scheme should inevitably be implicit, it can be non-iterative by using spacially factored tridiagonal matrix inversion scheme, and in most cases it is unconditionally stable. This method is especially effective when the original equation is expressed in conservation-law form. It is successfully applied to solve one-dimensional convection equation, Burgers equation, two-dimensional quasi-boundary layer equation and two-dimensional compressible Navier-Stokes equations.

Finite Difference Scheme

As an example, let us think of Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

To get a numerical solution of this equation, we discretize t and x as follows:

$$t_0=0, t_1=\Delta t, \cdots, t_n=n\Delta t, \cdots,$$

$$x_0=0, x_1=\Delta x, \cdots, x_j=j\Delta x, \cdots, x_\ell=\ell\Delta x(=1).$$

First, we rewrite the original equation into conservation law form:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}u^2-\nu\frac{\partial u}{\partial x}\right)=0. \tag{2}$$

Integrating the equation by time t and coordinate x in the domain (see Fig.1)

$$x_{j-1}\leqq x\leqq x_{j+1}, \quad t_n\leqq t\leqq t_{n+1},$$

we get

$$\int_{x_{j-1}}^{x_{j+1}}\int_{t_n}^{t_{n+1}}\frac{\partial u}{\partial t}dt\,dx+\int_{t_n}^{t_{n+1}}\int_{x_{j-1}}^{x_{j+1}}\frac{\partial}{\partial x}\left(\frac{1}{2}u^2-\nu\frac{\partial u}{\partial x}\right)dx\,dt. \tag{3}$$

This leads to

$$\int_{x_{j-1}}^{x_{j+1}}(u^{n+1}-u^n)dx+\int_{t_n}^{t_{n+1}}\left\{\frac{1}{2}u_{j+1}^2-\nu\left(\frac{\partial u}{\partial x}\right)_{j+1}-\frac{1}{2}u_{j-1}^2+\nu\left(\frac{\partial u}{\partial x}\right)_{j-1}\right\}dt, \tag{4}$$

where $u^n=u(x,t_n)$, $u_j=u(x_j,t)$ etc. Numerically we can use the values of u at discrete points (x_j, t_n) etc. Then we should approximate the integrations numerically. The integral

$$\int_{x_{j-1}}^{x_{j+1}} u^{n+1} dx \qquad (5)$$

is best approximated by Simpson rule because we know the values of u^{n+1} at $x = x_{j-1}, x_j, x_{j+1}$ only. Accordingly, this is approximated by

$$(u_{j-1}^{n+1} + 4u_j^{n+1} + u_{j+1}^{n+1})\frac{\Delta x}{3}. \qquad (6)$$

If above integral is simply approximated as

$$u_j^{n+1} \cdot 2\Delta x \qquad (7)$$

we can get a well-known Crank-Nicolson scheme for time integration. The integral

$$\int_{t_n}^{t_{n+1}} u_{j+1}^2 \, dt \qquad (8)$$

is best approximated by trapezoidal rule. That is

$$\left\{(u_{j+1}^n)^2 + (u_{j+1}^{n+1})^2\right\}\frac{\Delta t}{2} \qquad (9)$$

To approximate the integral

$$\int_{t_n}^{t_{n+1}} \left(\frac{\partial u}{\partial x}\right)_{j+1} dt, \qquad (10)$$

the differential quotient

$$\left(\frac{\partial u}{\partial x}\right)^{n+1}_{j+1} \tag{11}$$

should be approximated in our domain as

$$(3u^{n+1}_{j+1} - 4u^{n+1}_{j} + u^{n+1}_{j-1})/2\Delta x. \tag{12}$$

Then the integral is approximated by

$$(3u^{n}_{j+1} - 4u^{n}_{j} + u^{n}_{j-1} + 3u^{n+1}_{j+1} - 4u^{n+1}_{j} + u^{n+1}_{j-1})\frac{\Delta t}{\Delta x}. \tag{13}$$

Finally we can derive the finite difference analog for the original Burgers equation as follows:

$$\begin{aligned}(1-\rho_1 2\bar{u}^{n+1}-\rho_2)u^{n+1}_{i-1} + (4+2\rho_2)u^{n+1}_{i} + (1+\rho_1 2\bar{u}^{n+1}-\rho_2)u^{n+1}_{i+1} \\ = (1+\rho_1 2\bar{u}^{n}+\rho_2)u^{n}_{i-1} + (4-2\rho_2)u^{n}_{i} + (1-\rho_1 2\bar{u}^{n}+\rho_2)u^{n}_{i+1},\end{aligned} \tag{14}$$

where

$$\rho_1 = \frac{3}{4}\frac{\Delta t}{\Delta x},\ \rho_2 = \nu\frac{\Delta t}{\Delta x^2},\ \bar{u}^{n} = \frac{1}{2}(u^{n}_{i+1} + u^{n}_{i-1}).$$

We can easily see that the solution of this finite difference equation propagates along the approximate characteristics. This finite difference scheme is nonlinear and implicit. To solve it non-iteratively, some linearization is needed. The simplest way is to replace $\bar{u}^{n+1}$ to $\bar{u}^{n}$, this type of linearization is widely used. To increase the order of accuracy we can use

$$\bar{u}^{n+1} = 2\bar{u}^{n} - \bar{u}^{n-1} \tag{15}$$

It is easily proved that this scheme is unconditionally stable if ν is positive. Once this finite difference equation is linearized, we can solve it by tridiagonal matrix inversion; this can be done non-iteratively. If we use iterative scheme, we can evaluate $\overline{u}^{n+1}$ more accurately.

The same method can be applied to solve the Navier-Stokes equations. Combining this with Beam-Warming implicit non-iterative time-splitting method (1), we can construct a finite difference scheme for the hyperbolic equation

$$\frac{\partial u}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \tag{16}$$

as follows:

$$\begin{aligned}&\left(I+\frac{1}{6}\Delta x^2\frac{\partial^2}{\partial x^2}+\frac{\Delta t}{2}\frac{\partial}{\partial x}A^n\right)\left(I+\frac{1}{6}\Delta y^2\frac{\partial^2}{\partial y^2}+\frac{\Delta t}{2}\frac{\partial}{\partial y}B^n\right)v^n\\&=-\Delta t\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}\right)^n,\end{aligned} \tag{17}$$

where

$$v^n=u^{n+1}-u^n,\quad A=\frac{\partial F}{\partial u},\quad B=\frac{\partial G}{\partial u}.$$

It is shown that the newly added terms

$$\frac{1}{6}\Delta x^2\frac{\partial^2}{\partial x^2},\quad \frac{1}{6}\Delta y^2\frac{\partial^2}{\partial y^2} \tag{18}$$

play a very impotant role for the accuracy and it makes this scheme close to a kind of characteristic method.

Computed Results

Computed results for Burgers equation are shown in Figs.2-8. The initial condition is

$$u = \begin{cases} \sin 2\pi x & ; \quad 0 \leqq x \leqq 0.5 \\ 0 & ; \quad 0.5 \leqq x \leqq 1.0 \end{cases}$$

No artificial viscosity was used, but we could get a very clear shock with small shock-capturing oscillation. Moreover, it was found that a very small value of physical viscosity ν can suppress the oscillation very well, without affecting the solution before the shock wave is formed.

Conclusion

It was found that this integral formulation of fintie difference scheme for partial differential equation can give a very accurate and stable scheme. Although traditional formulation sometimes gives the same results, the method discribed here is very general and easily applied to any kind of partial differential equations without tedious cancelation of error terms.

Reference

1) Beam,R.M. and Warming,R.F.: An Implicit Finite-Difference Algorithm for Hyperbolic Systems in Conservation-Law Form, J.Comp.Phys. 22,87-110(1976).

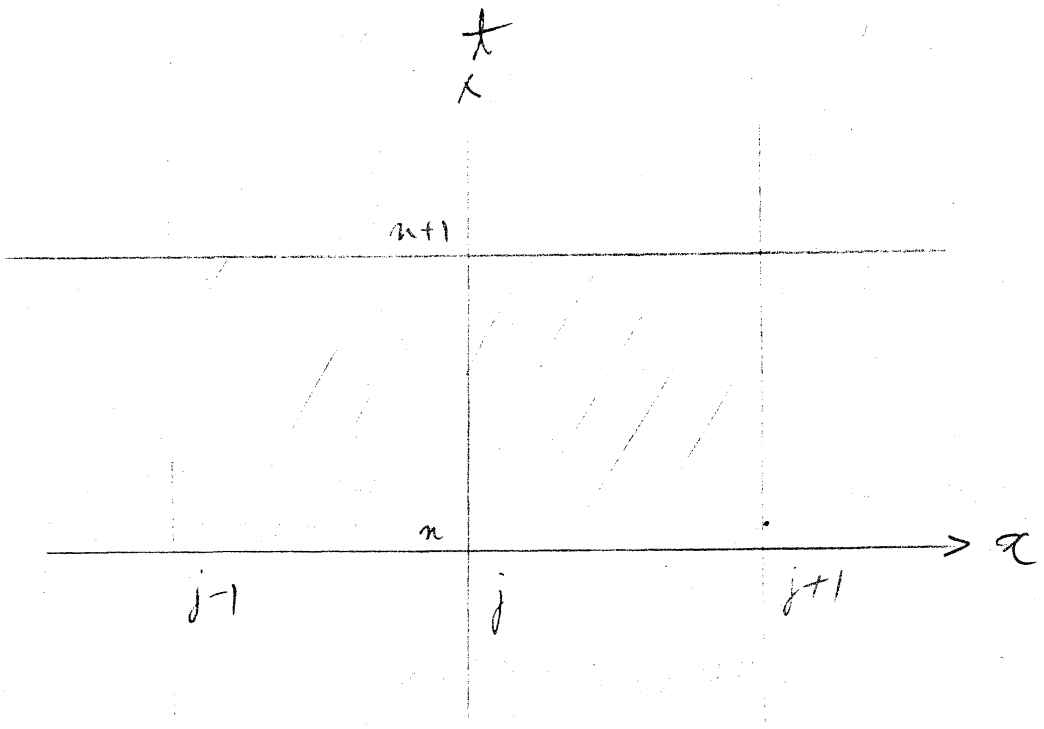
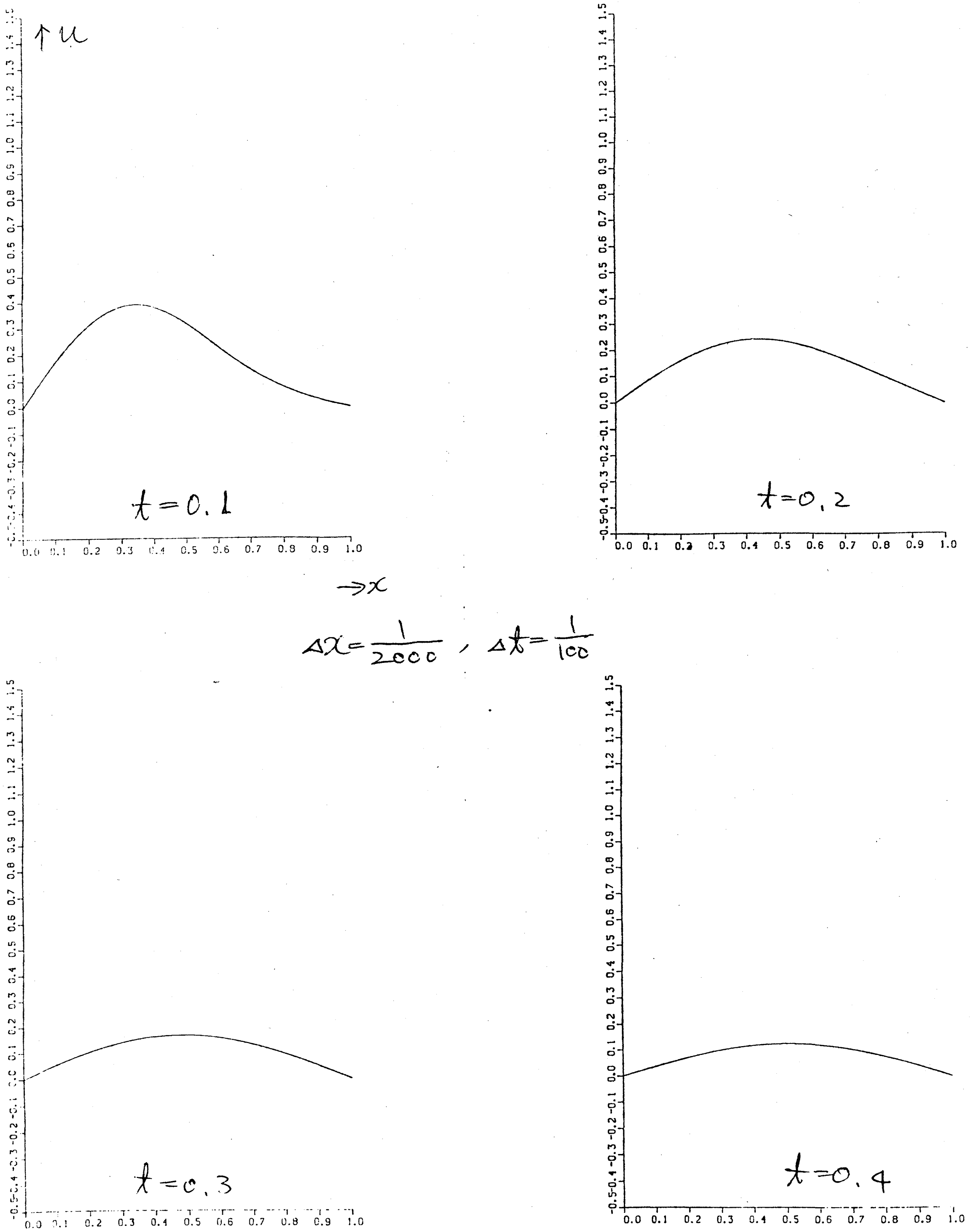


Fig. 1



$\Delta x = \frac{1}{2000}$, $\Delta t = \frac{1}{100}$

Fig. 2 $\nu = 1.0$

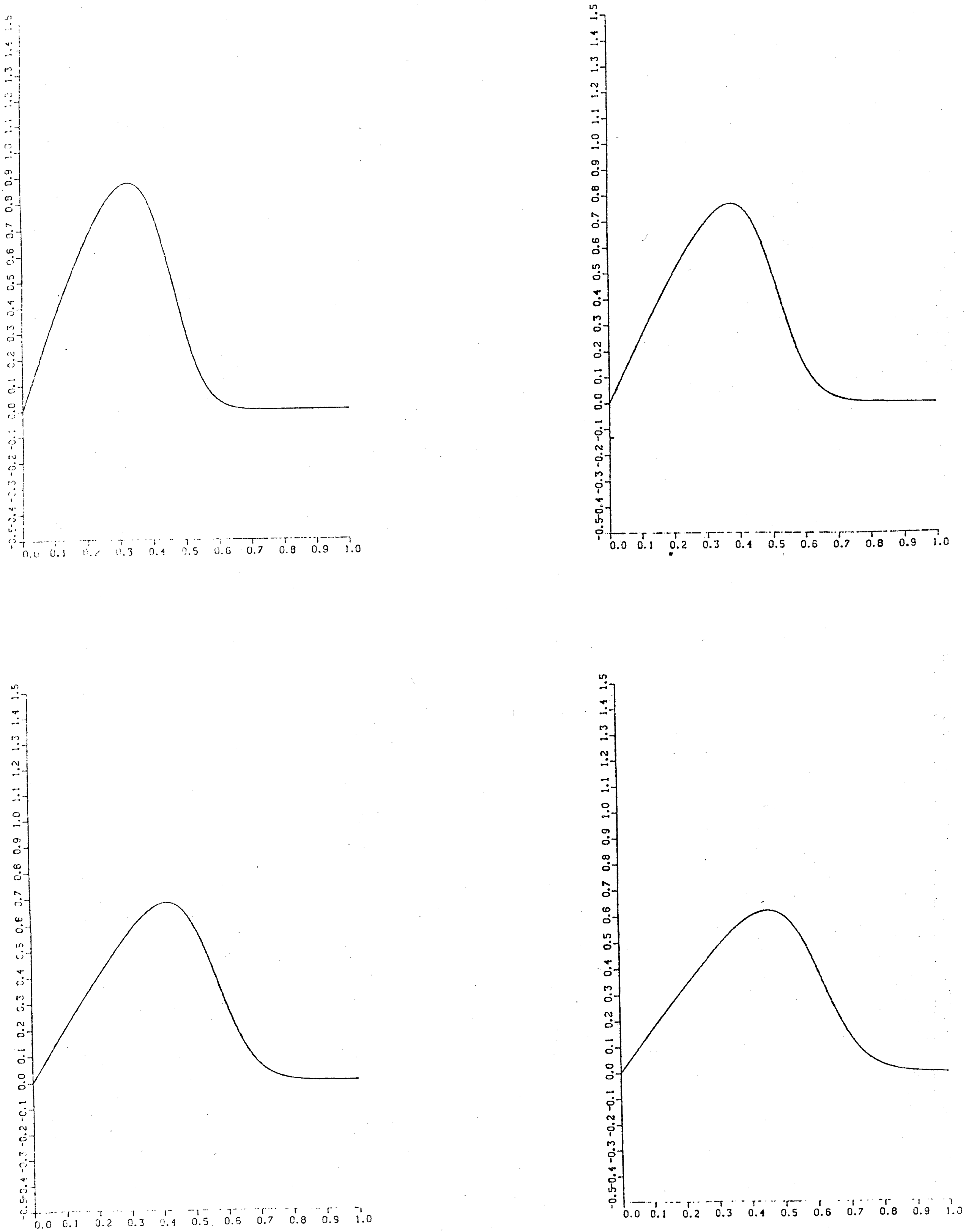


Fig. 3. $\nu = 1.0/10.0$

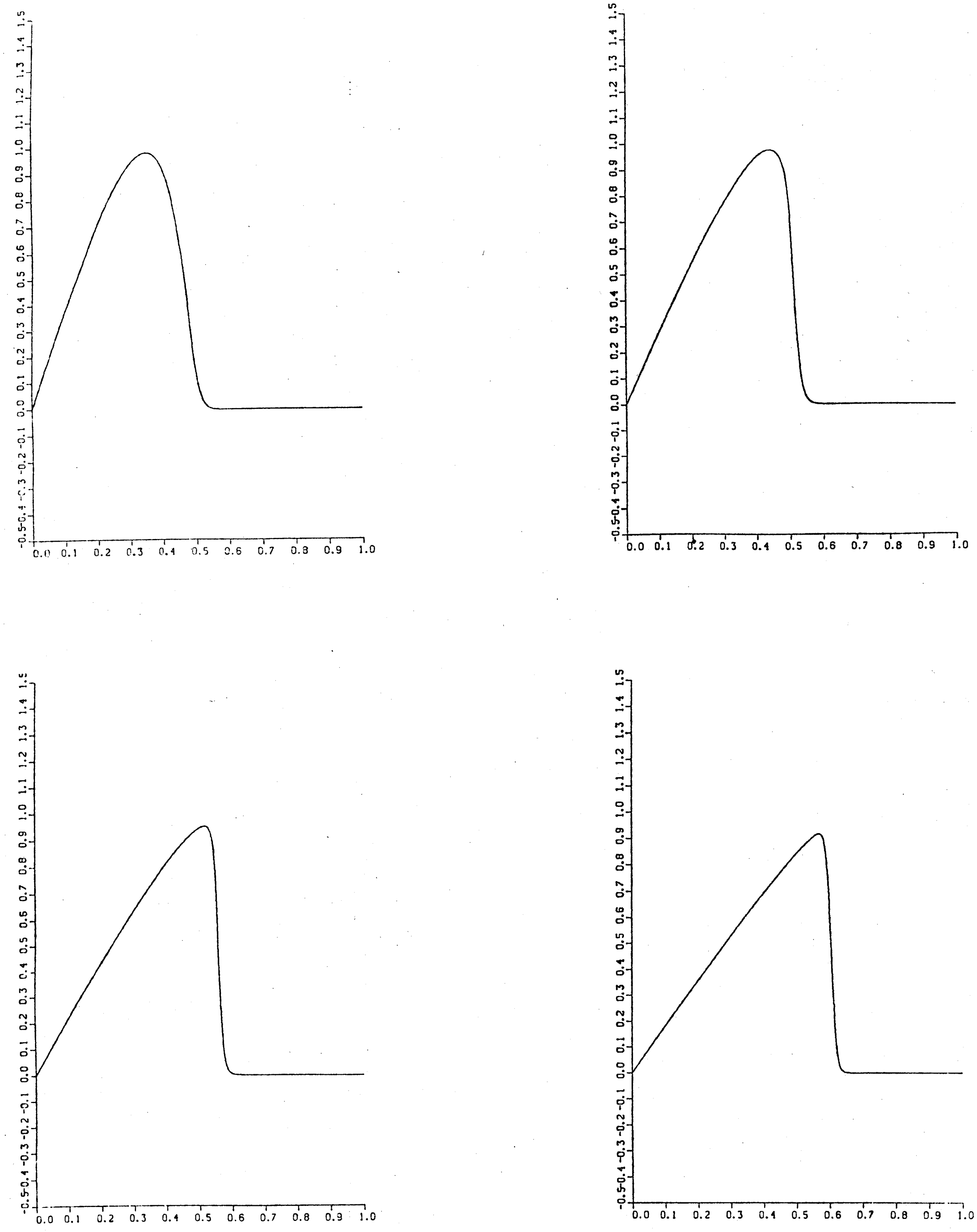


Fig. 4. $\nu = 1.0/100.0$

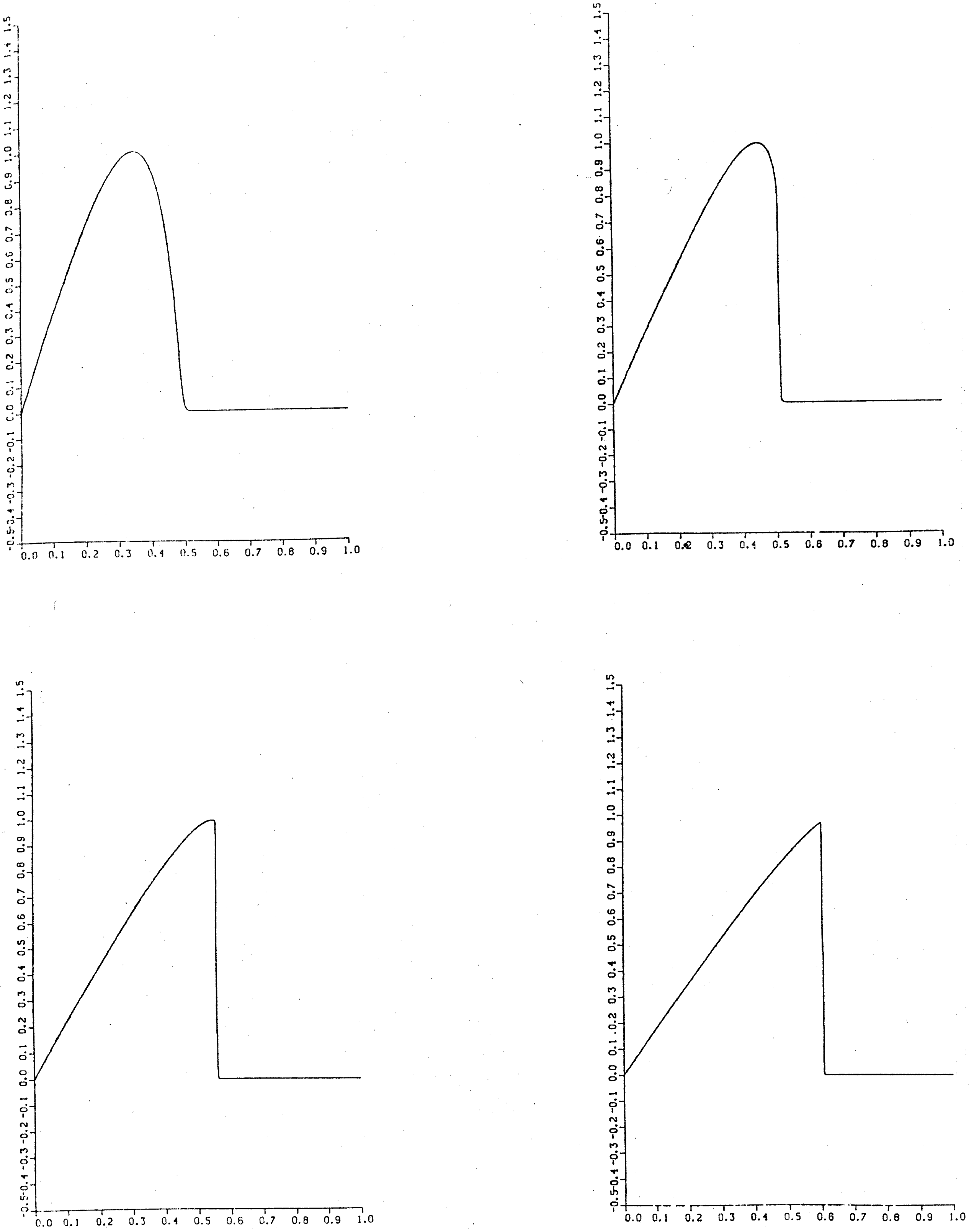


Fig. 5 $\nu = 1.0/1{,}000.0$

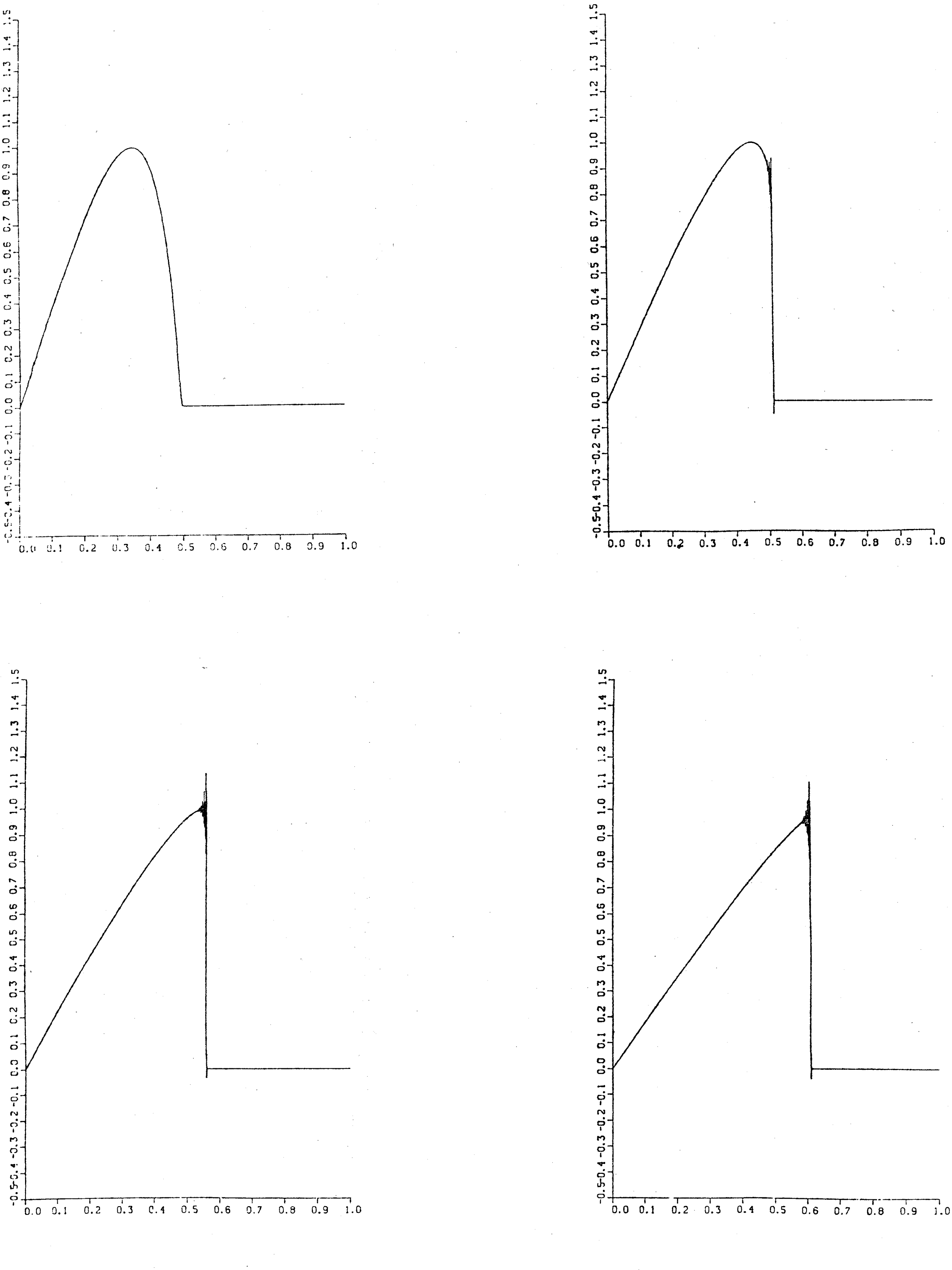


Fig. 6 $\nu = 1.0/10000.0$

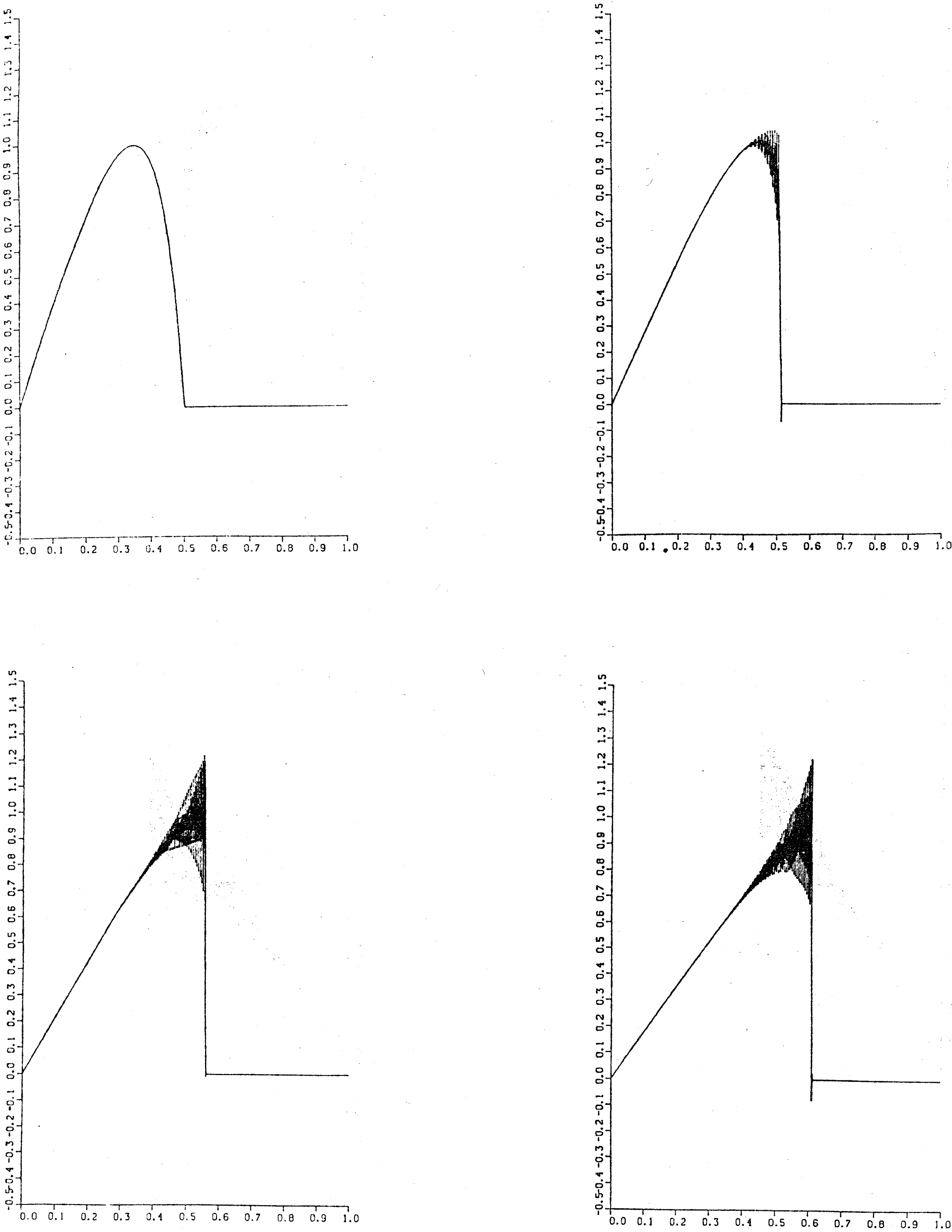


Fig.7 $\nu = 1.0/100000.0$

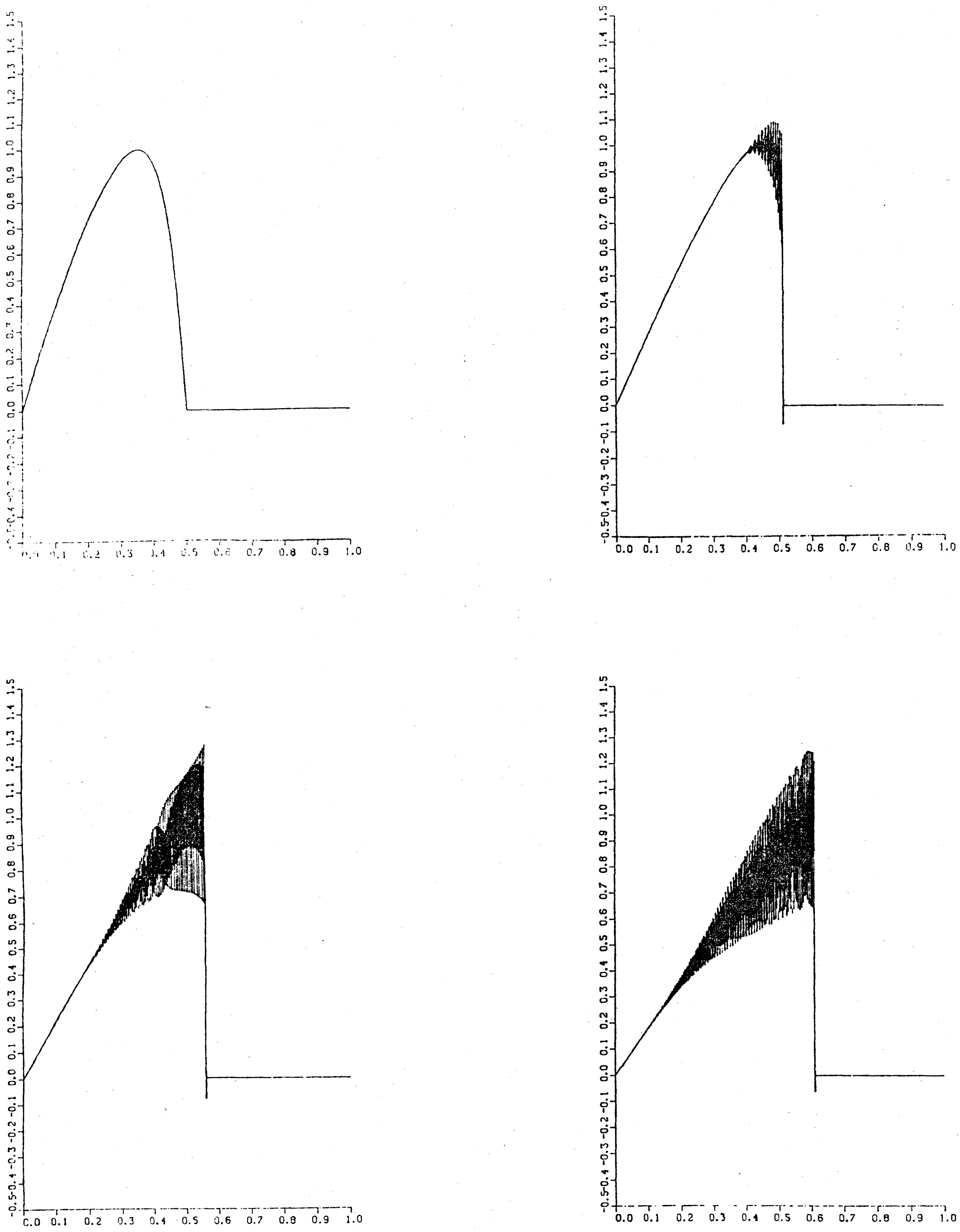


Fig. 8. $\nu = 1.0/1{,}000{,}000.0$